# Interval Graphs and (Normal Helly) Circular-arc Graphs

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Constrained Recognition Problems (ICALP 2018)

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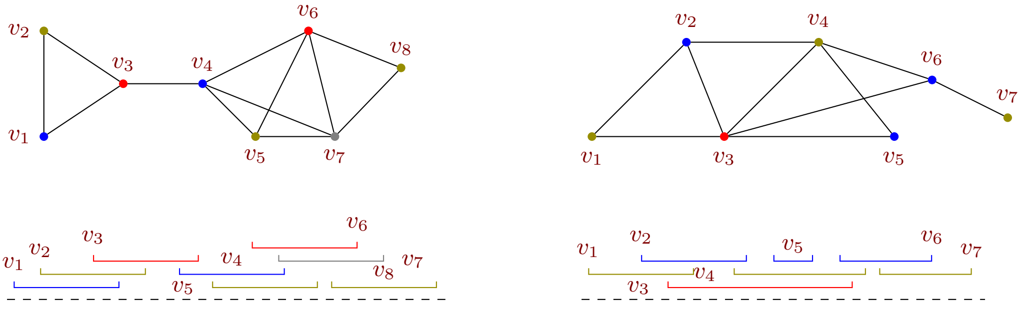
# Interval graphs




Interval graph:
There are a set $\mathcal{I}$ of intervals on the real line and $\phi : V(G) \to \mathcal{I}$ such that $uv \in E(G)$ if and only if $\phi(u)$ intersects $\phi(v)$.

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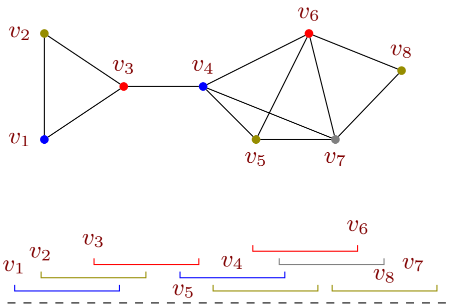


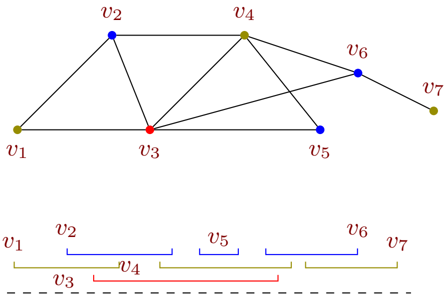


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If all the intervals have the same length, then it is a *unit interval graph*.

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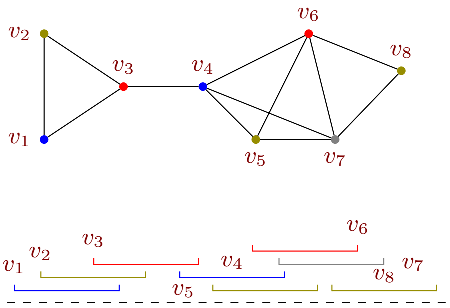


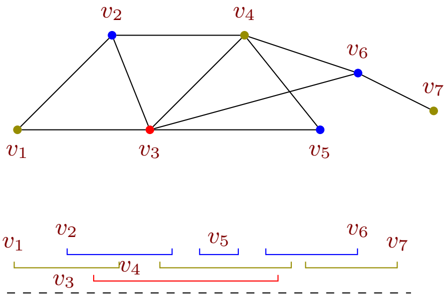


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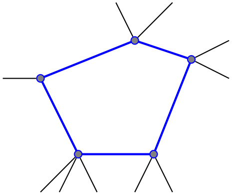
such that $uv \in E(G)$ if and only if $\phi(u)$ intersects $\phi(v)$.

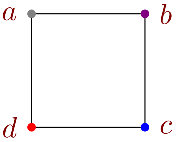
If all the intervals have the same length, then it is a *unit interval graph*.
If no interval is properly contained in another, then it is a *proper interval graph*.



A graph is *chordal* if it contains no holes.

INTERVAL $\subset$ CHORDAL

# Holes



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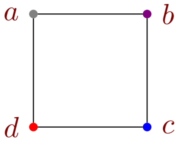


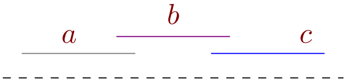


We cannot accommodate $d$.

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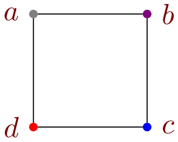


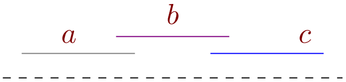


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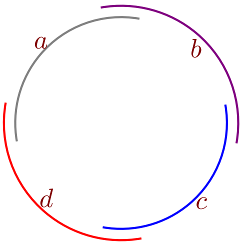


Circular-arc graph:
There are a set $\mathcal{A}$ of arc on a circle and $\phi : V(G) \to \mathcal{A}$ such that $uv \in E(G)$ iff $\phi(u)$ intersects $\phi(v)$.

INTERVAL $\subset$ CIRCULAR-ARC

$\textsc{interval} \subset \textsc{circular-arc}$

How about chordal circular-arc graphs?

$\textsc{interval} \subset \textsc{circular-arc}$

How about chordal circular-arc graphs?

Are they always interval graphs?

# The connection

# Pathologic intersecting patterns

Circular arcs admit some "pathologic" intersecting patterns not possible in intervals.

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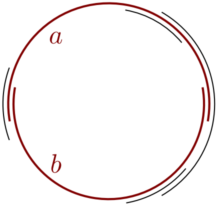


Pattern 1
two arcs intersect at both ends

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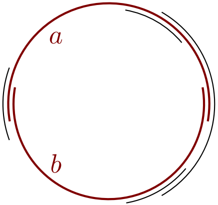


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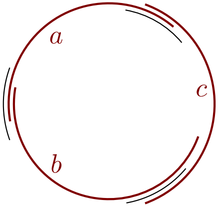


Pattern 2
three arcs pairwise intersect
w/o a common point

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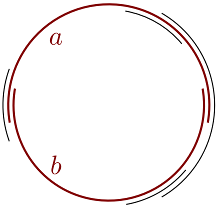


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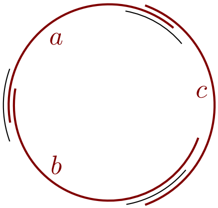


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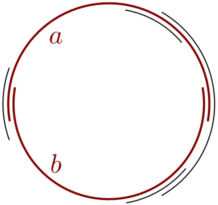


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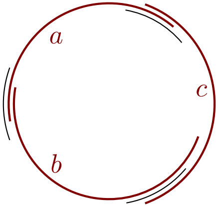


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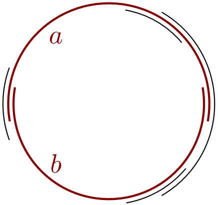


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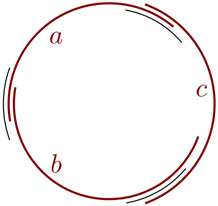


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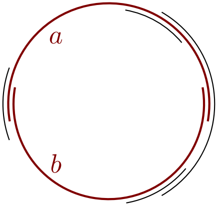


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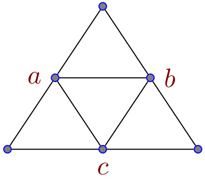


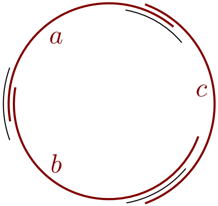


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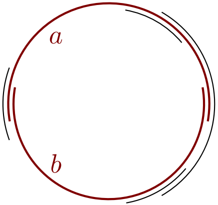


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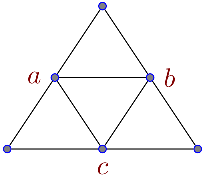


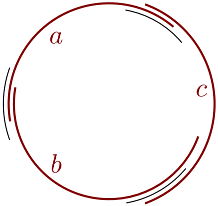


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normal circular-arc ∩ Helly circular-arc
≠ normal Helly circular-arc

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A graph is a *normal Helly circular-arc graph* if it has a model that is both normal and Helly.

[McKee 2003]
A circular-arc model is normal and Helly iff no $\leq 3$ arcs cover the whole circle.

(In other words, any minimal set of arcs covering the circle represents a hole).

Lemma. NORMAL HELLY CIRCULAR-ARC $\cap$ CHORDAL $=$ INTERVAL.

**Proof.**

Interval models are normal and Helly: INTERVAL $\subset$ NORMAL HELLY CIRCULAR-ARC.

In a normal Helly model $\mathcal{A}$ of a chordal graph $G$, there must be some point uncovered by any arc of $\mathcal{A}$. Thus, $\mathcal{A}$ is an interval model. □

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Lemma. $\textsc{Normal Helly circular-arc} \cap \textsc{chordal} = \textsc{interval}$.

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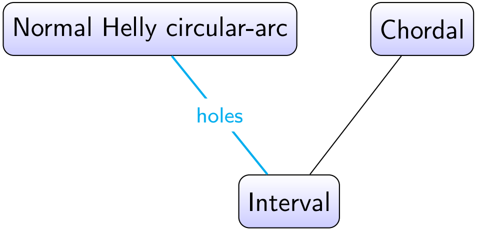
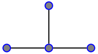
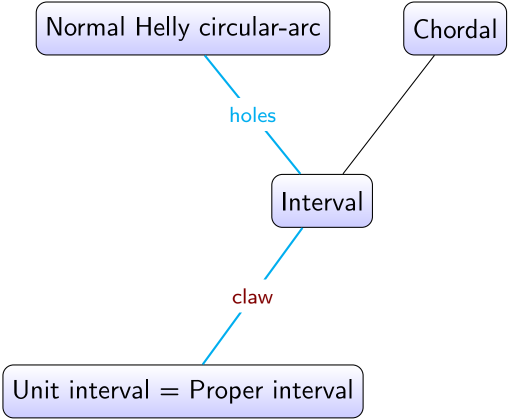



[Lin et al. 2013]; [C 2017]



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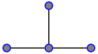

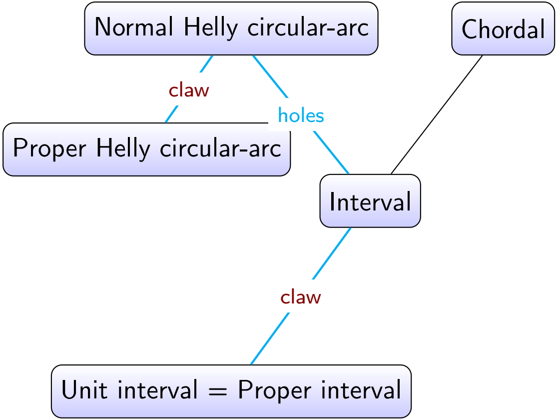


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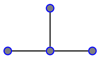

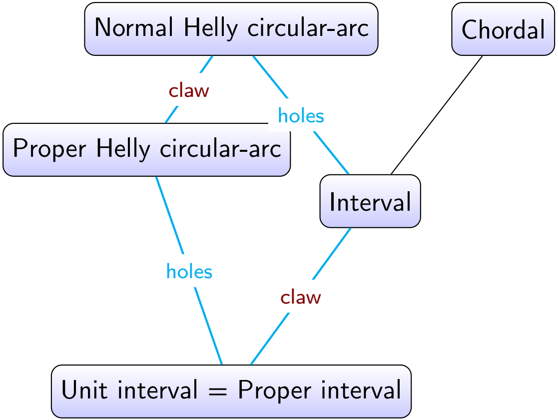


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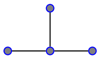

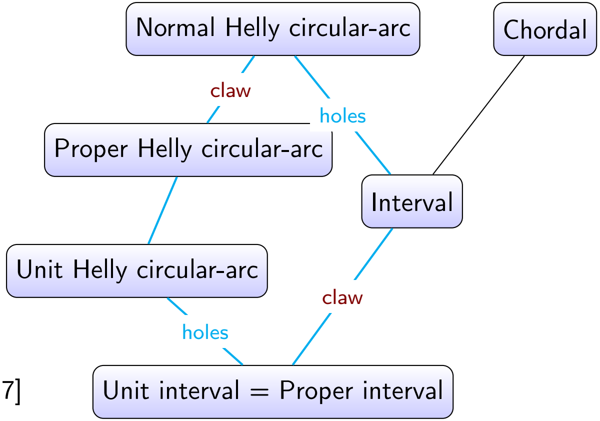


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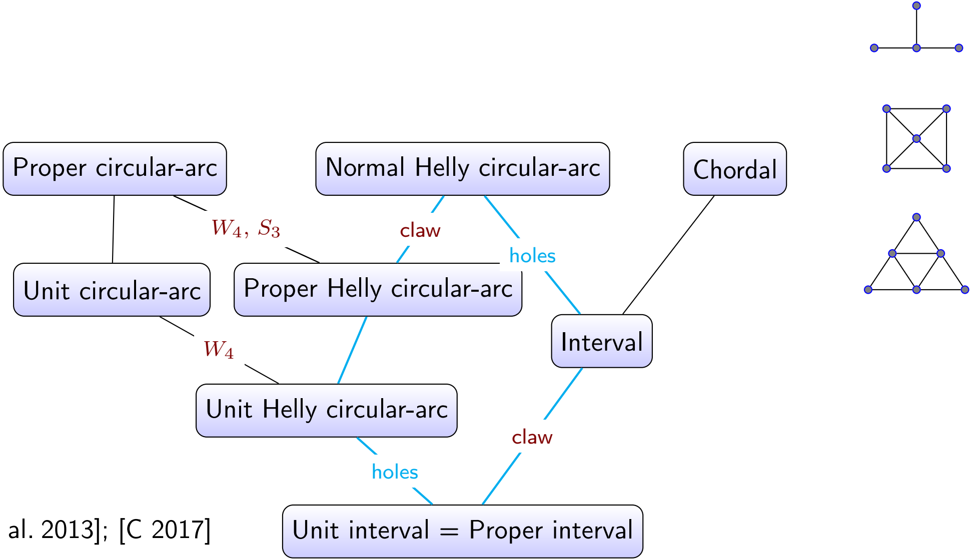


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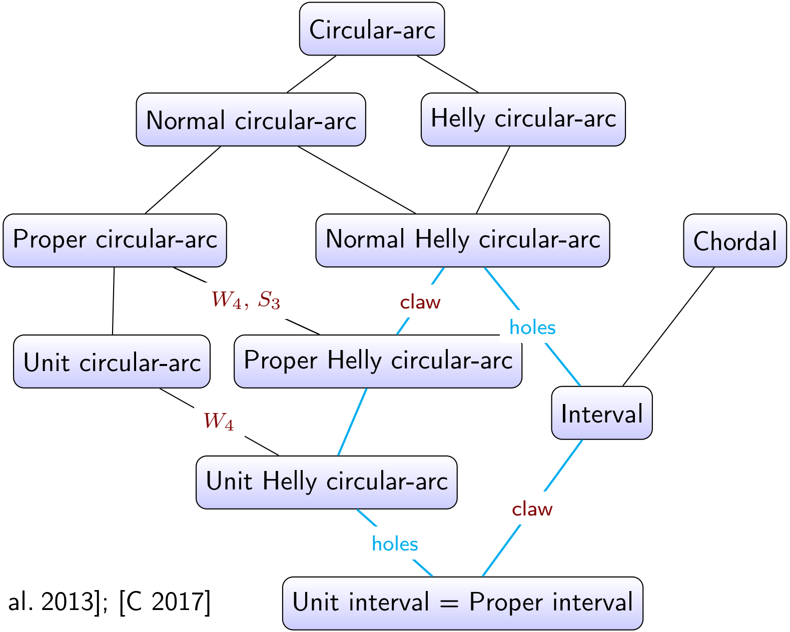


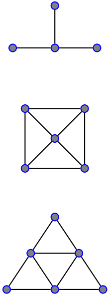

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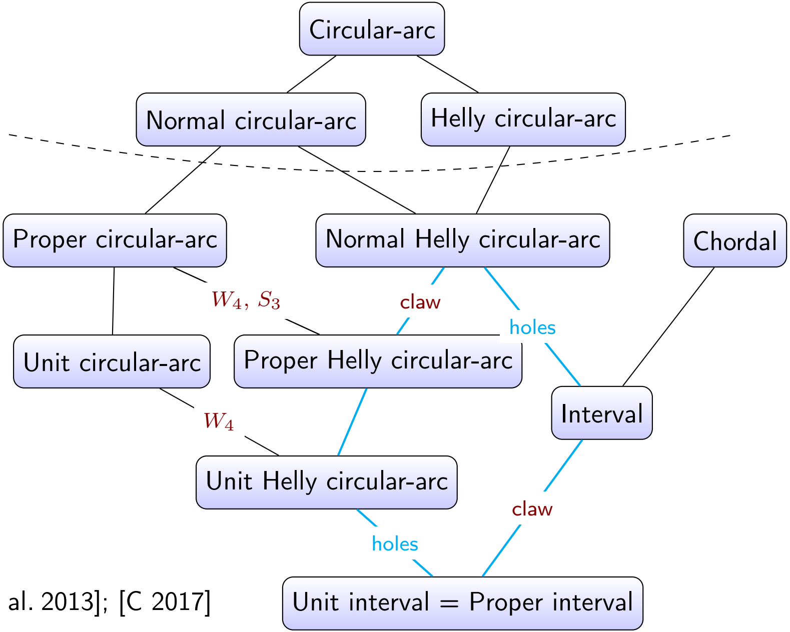


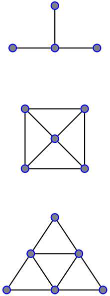

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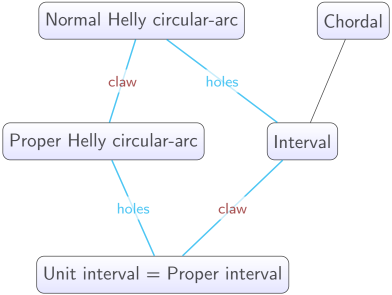
# Unit interval graphs



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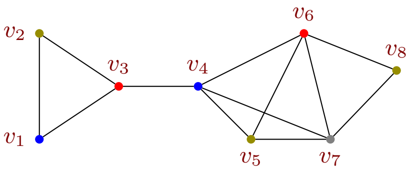


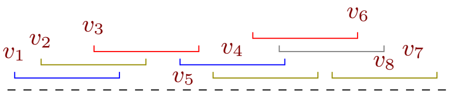

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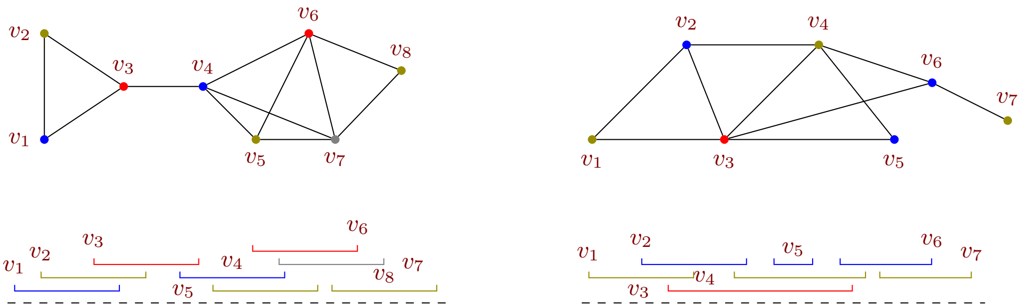
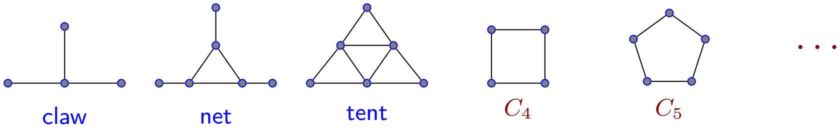



The left is a unit interval graph; the right is not.

# Forbidden induced subgraphs

[Wegner 1967]




UNIT INTERVAL $\subset$ INTERVAL $\subset$ CHORDAL

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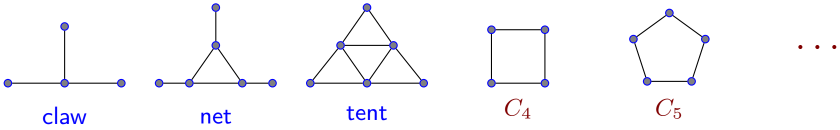


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# Unit interval vertex deletion

**Unit interval vertex deletion**

*Input:* A graph $G$ and an integer $k$.
*Task:* A set $V_-$ of $\leq k$ vertices such that $G - V_-$ is a unit interval graph.

NP-complete

[Lewis & Yannakakis 1978]

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| FPT | $O((14k + 14)^{k+1} \cdot kn^6)$ | $O(6^k \cdot n^6)$ | $O(6^k \cdot (n + m))$ |
|---|---|---|---|
| [Marx 2006] | [van Bevern et al. 2010] | [Villanger 2013] | [C 2017] |

# Main ideas

**Standard technique**

A small subgraph $F$ can be found in $n^{|F|}$ time and dealt with an $|F|$-way branching.

Make it $\{\text{claw}, \text{net}, \text{tent}\}$-free, then solve it using chordal vertex deletion

[van Bevern et al. 2010]
Make it $\{\text{claw}, \text{net}, \text{tent}, C_4, C_5, C_6\}$-free, and then use iterative compression.

[Villanger 2013]
A connected $\{\text{claw}, \text{net}, \text{tent}, C_4, C_5, C_6\}$-free graphs are proper circular-arc graphs, on which the problem can be solved in linear time.
(by manually building a proper circular-arc model.)

# Proper Helly circular-arc graphs

A graph having a circular-arc model that is *both proper and Helly*.

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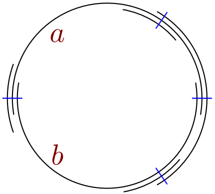
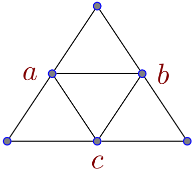
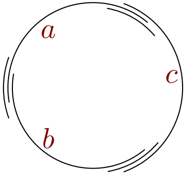



A Helly model



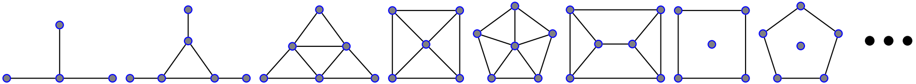




A proper model

# Why proper Helly?

**Theorem (Tucker 1974; Lin et al. 2013)**

*A graph is a proper Helly circular-arc graph if and only if it contains no claw, net, tent, $W_4$, $W_5$, $\overline{C_6}$, or $C_\ell^*$ for $\ell \geq 4$ (a hole $C_\ell$ and another isolated vertex).*



A trivial corollary:
If a proper Helly circular-arc graph is chordal, then it is a unit interval graph.

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A connected $\{\text{claw}, \text{net}, \text{tent}, C_4, C_5\}$-free graph is a proper Helly circular-arc graph.

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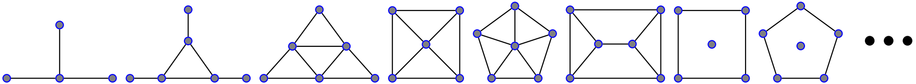

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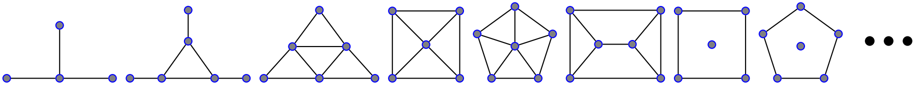
# Why proper Helly?

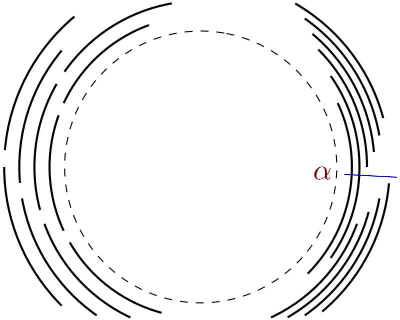
**Theorem (Tucker 1974; Lin et al. 2013)**

*A graph is a proper Helly circular-arc graph if and only if it contains no claw, net, tent, $W_4$, $W_5$, $\overline{C_6}$, or $C_\ell^*$ for $\ell \geq 4$ (a hole $C_\ell$ and another isolated vertex).*



A trivial corollary:
If a proper Helly circular-arc graph is chordal, then it is a unit interval graph.

A nontrivial corollary:
A connected $\{\text{claw}, \text{net}, \text{tent}, C_4, C_5\}$-free graph is a proper Helly circular-arc graph.

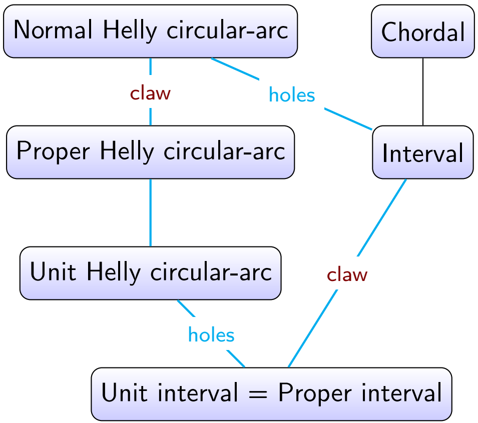



Once all claws, nets, tents, $C_4$'s, and $C_5$'s destroyed, it suffices to find the thinnest point from the model.

## Break time

You may safely skip the following three slides if you are tired.

# How about unit Helly circular-arc graphs?



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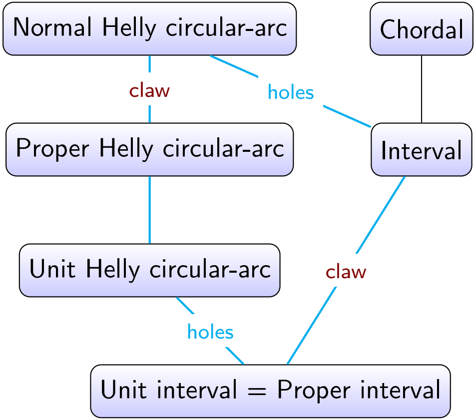


= PROPER HELLY CIRCULAR-ARC ∩ CHORDAL

= UNIT HELLY CIRCULAR-ARC ∩ CHORDAL

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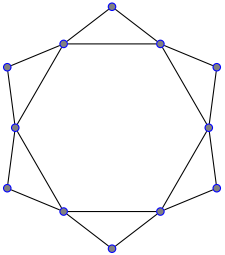

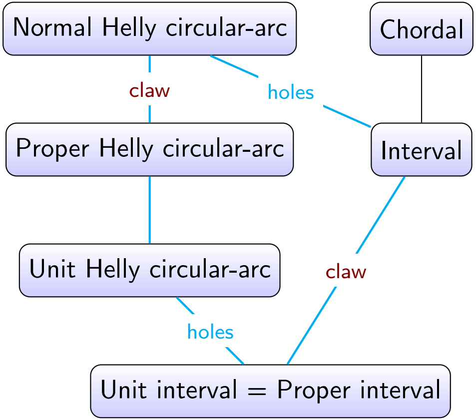


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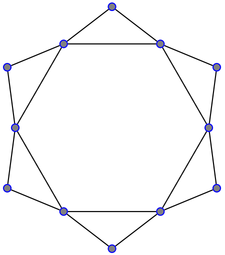
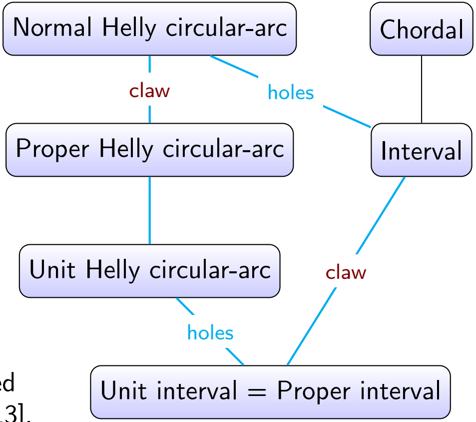


This is actually the $\mathsf{CI}(\ell, 1)$ graph defined by [Tucker 1974]; see also [Lin et al. 2013].




= PROPER HELLY CIRCULAR-ARC $\cap$ CHORDAL
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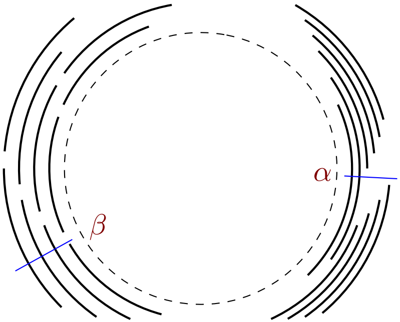
# Edge deletion

proper Helly circular-arc $\rightarrow$ unit interval by deleting edges:
Achilles' heel with respect to edges.

The thinnest point for vertices is $\alpha$
The thinnest point for edges is $\beta$



# A slightly stronger statement

[van Bevern et al. 2010]
Unit interval vertex deletion remains NP-hard on {claw, net, tent}-free graphs.
[Villanger 2013]
Unit interval vertex deletion is in P for {claw, net, tent, $C_4$, $C_5$, $C_6$}-free graph.

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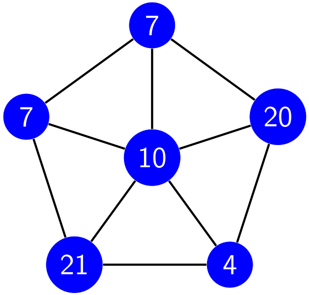


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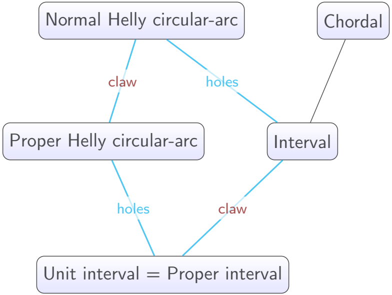
# Normal Helly circular-arc graphs



# The problems

Characterization (by forbidden induced subgraphs):
Identify the set $\mathcal{H}$ of minimal subgraphs such that $G$ is a normal Helly circular-arc graphs if and only if it contains no subgraph in $\mathcal{H}$.

Recognition:
Efficiently decide whether a given graph is a normal Helly circular-arc graph or not.

Detection:
Either a model that is both normal and Helly (positive certificate),
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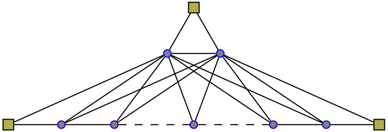
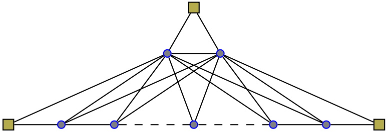


Asteroidal triple (AT):
Three vertices of which each pair is connected by a path avoiding neighbors of the third one.

# Characterization of interval graphs



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**Theorem (Lekkerkerker and Boland, 1962)**

*A graph is an interval graph if and only if it contains no holes or ATs.*

any hole of length $\geq 6$ contains ATs.

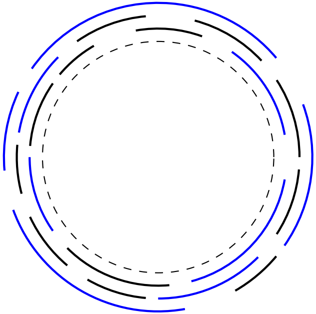
# Chordal asteroidal witnesses (CAW)

Asteroidal witness: a minimal graph that contains an AT.

All chordal asteroidal witnesses are minimal forbidden induced subgraphs of NHCAG.
(Recall that NORMAL HELLY CIRCULAR-ARC $\cap$ CHORDAL $=$ INTERVAL.)

We are henceforth focused on the non-chordal case, hence holes.

# Intuition



In a normal Helly circular-arc model,

- Any minimal set of arcs covering the circle induces a hole.
- For any vertex $v$ in a hole,

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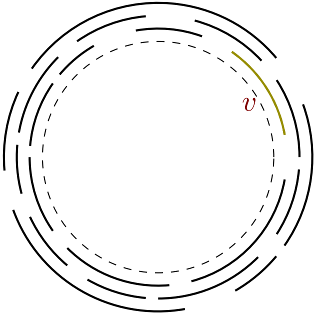


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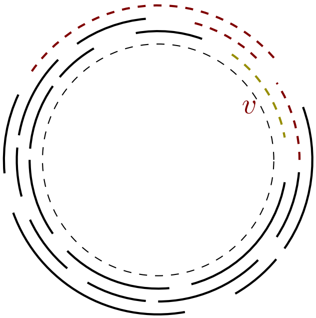


In a normal Helly circular-arc model,

- Any minimal set of arcs covering the circle induces a hole.
- For any vertex $v$ in a hole, $G - N[v]$ is an interval subgraph.

Construction of $\mho(G)$:$^{\star}$

1. find a vertex $v$ with the largest degree;
   ($G - N[v]$ is an interval graph.)
2. append a copy of $N[v]$ to "each end" of $G - N[v]$.
3. add a new vertex $w$ to keep the left end of the left copy of $N[v]$.

$\star$: Upon a failure during this construction, a forbidden induced subgraph can be detected.

# The auxiliary graph $\mho(G)$

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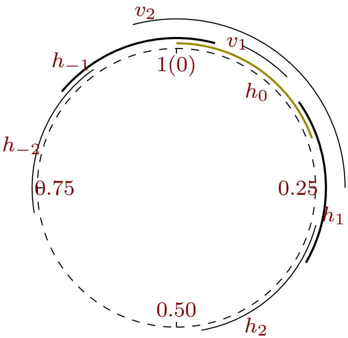
**Theorem (C 2016; C Grippo Safe 2017)**

*$G$ is a normal Helly circular-arc graph if and only if $\mho(G)$ is an interval graph.*

# Circular-arc model for $G \Rightarrow$ Interval model for $\mho(G)$




every point in the model has a value in $(0, 1]$.

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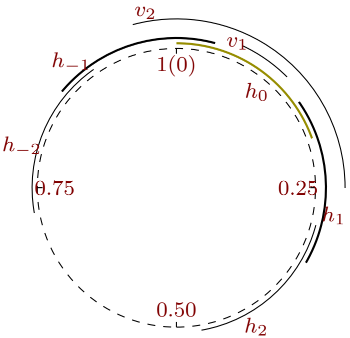


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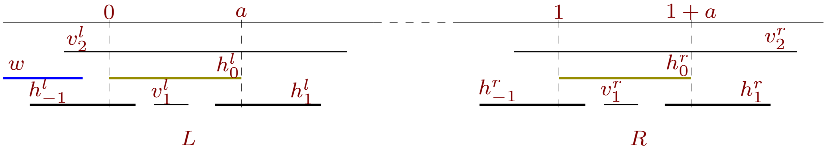
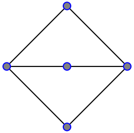
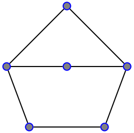


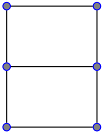
# Other forbidden induced subgraphs (with holes)



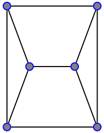
$K_{2,3}$



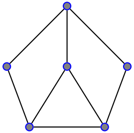
twin-$C_5$

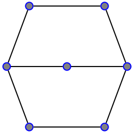


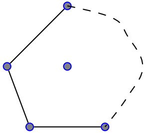
domino



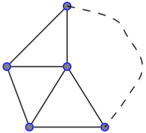
$\overline{C_6}$







$C^*$



wheel

# The certifying recognition algorithm

1. **if** $G$ is chordal **then**
   **return** an interval model of $G$ or a CAW;
2. build the auxiliary graph $\mho(G)$;
3. **if** $\mho(G)$ is an interval graph **then**
   build a normal and Helly circular-arc model $\mathcal{A}$ for $G$;
   **return** $\mathcal{A}$;
4. **else**
   find a minimal forbidden induced subgraph $F$ of $G$;
   **return** $F$.

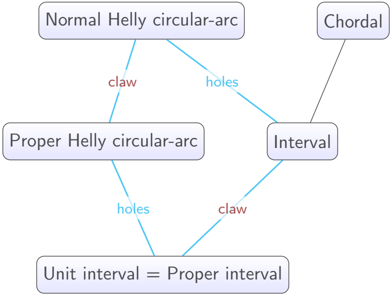
# Related subclasses of circular-arc graphs

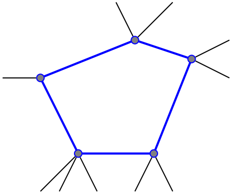
| | Characterization | Certifying recognition |
|---|---|---|
| CIRCULAR ARC (CA) | Unknown | Unknown† |
| NORMAL CA | Unknown‡ | Unknown |
| PROPER CA | Tucker 1974 | Kaplan&Nussbaum 2009 |
| UNIT CA | Tucker 1974 | Kaplan&Nussbaum 2009 |
| UNIT HELLY CA | Lin et al. 2013 | Lin et al. 2013 |
| PROPER HELLY CA | Lin et al. 2013 | Lin et al. 2013 |
| NORMAL HELLY CA | C Grippo & Safe 2017 | |

†: linear recognition is known.

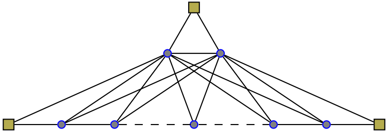
‡: circular arc graphs that are not normal are known.

# Interval graphs





Hole:
an induced cycle of length $\geq 4$.



Asteroidal triple (AT):
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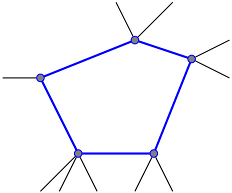
# Characterization of interval graphs



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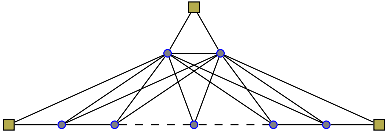

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NORMAL HELLY CIRCULAR-ARC $\cap$ CHORDAL $=$ INTERVAL.

# Reduction: small forbidden subgraphs

Recall that

**Standard technique**

A small subgraph $F$ can be found in $n^{|F|}$ time and dealt with an $|F|$-way branching.

Kill all forbidden subgraphs of $\leq 10$ vertices: The resulting graph is called *reduced*.

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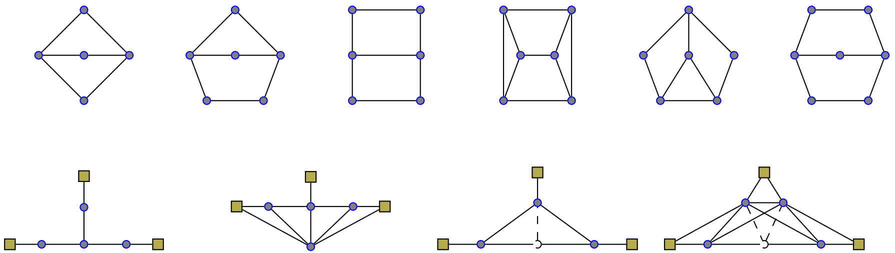
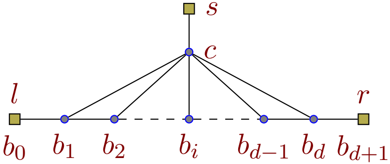
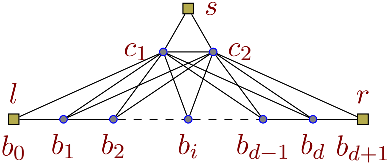


We are left with long holes (at least 11 vertices) and




Shallow terminal:
of the unique asteroidal triple, one vertex $s$ has a shorter distance to the other two $(l, r)$.

# Main theorem

In a reduced graph,

- shallow terminals form modules (set of vertices with the same neighborhood); and
- neighbors of each of the modules induces a clique.

Or (in the parlance of modular decomposition):
Each shallow terminal in the quotient graph of a reduced graph is simplicial.

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# Maximal cliques

Shallow terminals are not in any holes; the rest form a normal Helly circular-arc graph.
$n$ maximal cliques
chordal graph: tree interval graph: path normal Helly circular-arc graph: cycle reduced graph: olive ring.

# Linear-time

# Almost interval graphs

**Theorem (Yannakakis 79, 81; Goldberg et al. 95)**

*All modification problems to interval graphs are NP-complete.*

- interval $+ ke$, interval $- ke$, and interval $+ kv$ can be recognized in time $n^{O(k)}$ (polynomial for fixed $k$) [trivial].
- interval $- ke$ can be recognized in time $k^{2k} \cdot n^5$: [Heggernes et al. STOC'07]; and interval $+ kv$ can be recognized in time $k^9 \cdot n^9$ [Cao & Marx SODA'14].

$f(k) \cdot n^{O(1)}$: Fixed-parameter tractable (FPT)

- Can interval $+ ke$ be recognized in FPT time as well?
- Can any of them be recognized in linear time?

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# Prime graphs

**Definition**

- $M \subseteq V(G)$ is a *module* of $G$ if they have the same neighborhood outside $M$: $u, v \in M$ and $x \notin M$, $u \sim x$ iff $v \sim x$.
- A graph $G$ is prime if a module of $G$ is $V(G)$ or consists of a single vertex.

**Observation** details omitted

It suffices to solve the problem on prime graphs.

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# Conclusion

- We have used the connection as a black box to devise a $10^k \cdot n^{O(1)}$-time algorithm for the interval vertex deletion problem.
- Using it as a white box, the runtime can be improved to $O(10^k \cdot (n + m))$.
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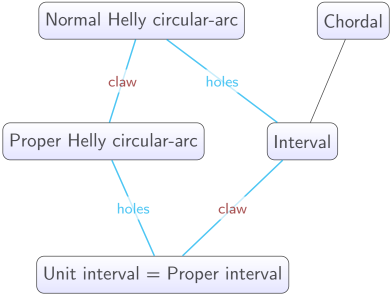
Normal Helly circular-arc
Chordal
claw
holes
Proper Helly circular-arc
Interval
holes
claw
Unit interval = Proper interval

# Epilogue

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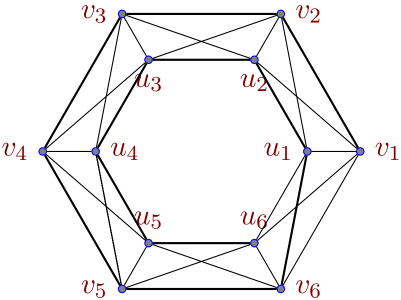

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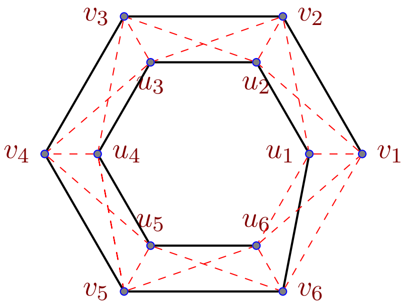
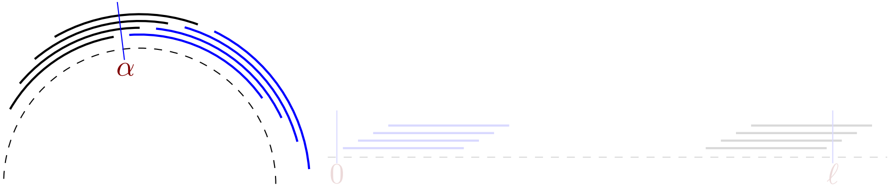


# To break long holes

## Definition

$$\overrightarrow{E}(\alpha) = \{vu : \alpha \in A_v,\ \alpha \notin A_u, v \to u\},$$

where $v \to u$ means that arc $A_v$ intersects arc $A_u$ from the left.




## A trivial corollary

For any point $\alpha$, the subgraph $G - \overrightarrow{E}(\alpha)$ is a unit interval graph.

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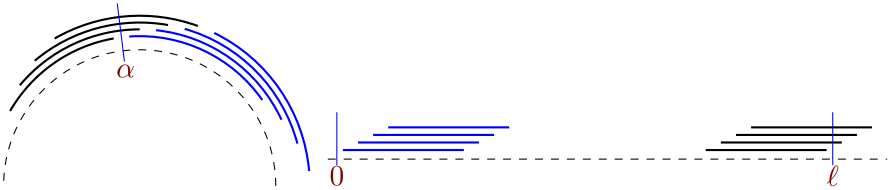


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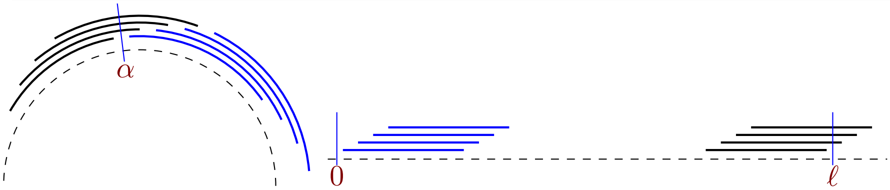


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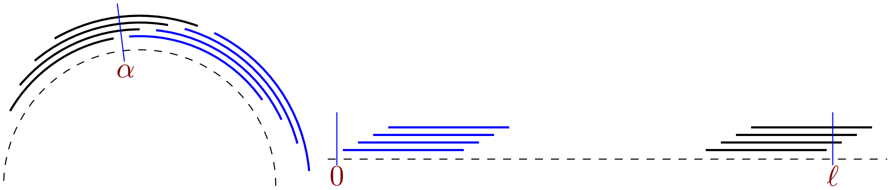
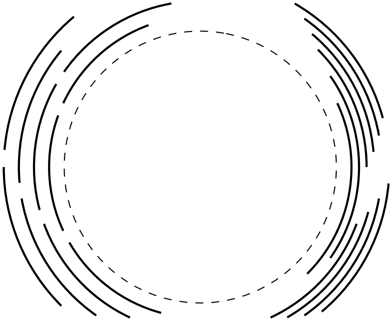



## A nontrivial corollary

Any minimum solution is $\overrightarrow{E}(\alpha)$ for some point $\alpha$.

# To find Achilles' heel

- Both deletion problems reduce to find a weakest point.
- A weakest point w.r.t. edges is not necessarily a weakest point w.r.t. vertices.



- it suffices to try $2n$ different points ($n$ actually).
- finding an arbitrary point $\rho$ and calculate $\overrightarrow{E}(\rho)$.
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**Theorem**

*both unit interval vertex deletion and unit interval edge deletion can be solved in $O(m)$ time on proper Helly circular-arc graphs.*

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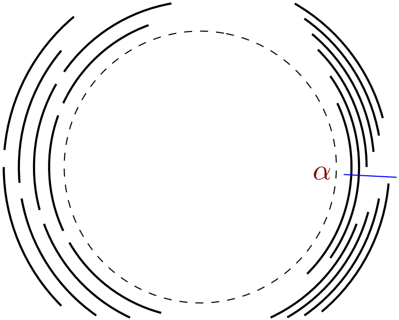


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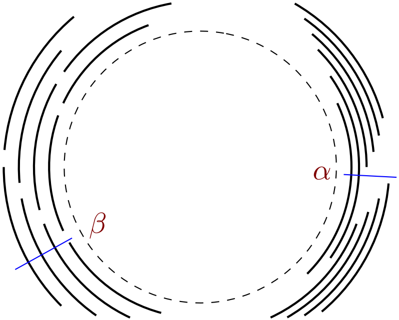


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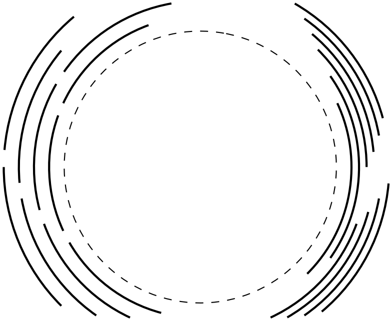

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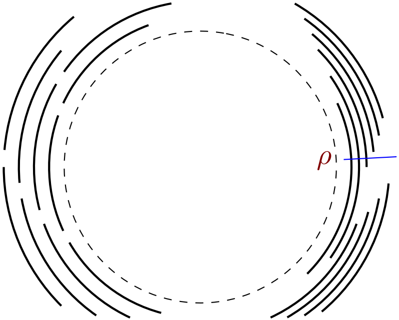


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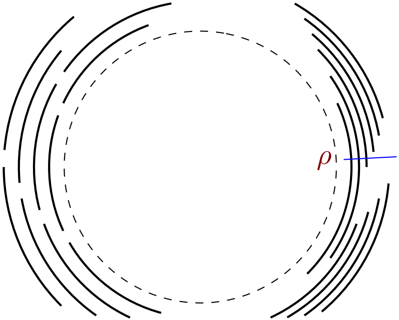


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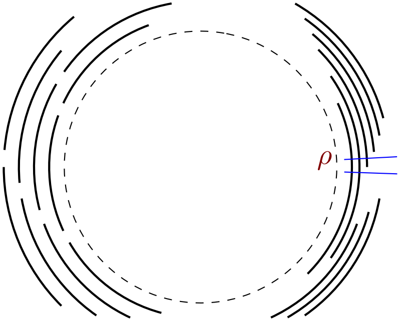


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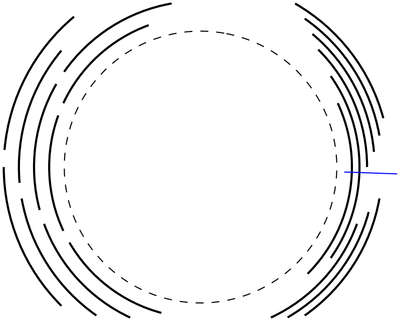

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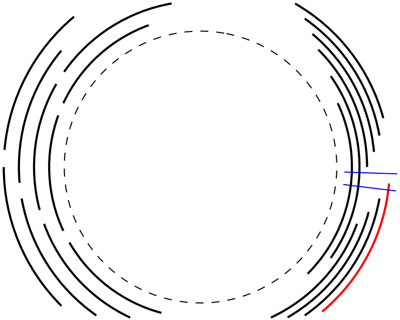

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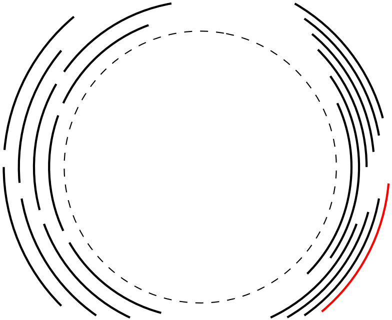

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